

Complex supermanifolds of low odd dimension and the example of the complex projective line

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Abstract

Complex supermanifold structures being deformations of the exterior algebra of a holomorphic vector bundle, have been parametrized by orbits of a group on non-abelian cohomology (see [5]). For the case of odd dimension 4 and 5 an identification of these cohomologies with a subset of abelian cohomologies being computable with less effort, is provided in this article. Furthermore for a rank ≤ 3 sub vector bundle $F \rightarrow M$ of a holomorphic vector bundle $E = F \oplus F' \rightarrow M$, a reduction of a (possibly non-split) supermanifold structure associated with ΛE to a structure associated with ΛF is defined. In the case of $rk(F') \leq 2$ with no global derivations increasing the \mathbb{Z} -degree by 2, the complete cohomological information of a supermanifold structure associated with E is given in terms of cohomologies compatible with the decomposition of E . Details on supermanifold structures of odd dimension 3 and 4 associated with sums of line bundles of sufficient negativity on $\mathbb{P}^1(\mathbb{C})$ are deduced.

Complex non-split supermanifolds arise as deformations of a split complex supermanifold $(M, \mathcal{O}_{\Lambda E})$ constructed from a complex vector bundle $E \rightarrow M$. They are parametrized by orbits of the group of bundle automorphism $H^0(M, Aut(E))$ on a certain in general non-abelian cohomology $H^1(M, G_E)$ (see [5]). The cochains of this cohomology can be expressed as the exponential of elements in a certain abelian cochain complex $C^1(M, Der^{(2)}(\Lambda E))$ (see [8]). However the degree of the involved finite exponential series is k for E of rank $2k$ or $2k + 1$, increasing the complexity of computations for every second step in odd dimension. In particular $H^1(M, G_E)$ is zero up to odd dimension 1, abelian up to odd dimension 3 and in general non-abelian beyond this limit.

A method for relating supermanifolds of higher odd dimension to abelian cohomologies can hence considerably simplify computations. We restrict our view to the case of no global second degree derivations, i.e. $H^0(M, Der_2(\Lambda E)) = 0$. The main object of interest in this article is relating the lowest dimensional non-abelian cases of odd dimension 4 and 5 to the abelian case. For this the non-abelian cohomology classifying supermanifold structures of odd dimension 4 or 5 is embedded as a subset into the abelian cohomology $H^1(M, Der^{(2)}(\Lambda E))$ (second section). This inclusion depends on a fixed map D relating cochains with values in a subsheaf $Der_2(\Lambda E) \subset Der^{(2)}(\Lambda E)$ that are appropriate for building supermanifolds, to cochains with values in a transversal complement $Der_4(\Lambda E) \subset Der^{(2)}(\Lambda E)$. However the

image of the inclusion and the $H^0(M, \text{Aut}(E))$ -action on it do not depend on the choice of D . A different way of relating elements in $H^1(M, G_E)$ to cochains in an abelian complex was established in [7] using a smooth Hermitian metric on E and Hodge theory.

Furthermore it is proved that a reduction of the odd dimension is in general well defined for any subbundle $F \subset E$ of rank ≤ 3 with complement, i.e. $E = F \oplus F'$. For the case $\text{rk}(F') \leq 2$ the cohomological data for a classification of supermanifold structures is given in terms of a sum of abelian cohomologies defined compatibly with the decomposition of E (third section). This decomposition of cohomologies is of good use for the analysis of the $H^0(M, \text{Aut}(E))$ -orbit structure on $H^1(M, G_E)$ since it is preserved by $H^0(M, \text{Aut}(F) \times \text{Aut}(F'))$.

Details on the orbit structure on $H^1(M, G_E)$ with respect to a maximal compact subgroup of $H^0(M, \text{Aut}(E))$ are deduced for rank 3 and 4 vector bundles being sums of line bundles of sufficient negativity on $\mathbb{P}^1(\mathbb{C})$ (fourth section). Parameter spaces for special examples of supermanifold structures of odd dimension 3 and 4 on $\mathbb{P}^1(\mathbb{C})$ were discussed and classified before in [1], [2], [3], [4], [9], and [10]. The case of odd dimension 2 can be found in [10].

1 Non-Split supermanifold structures

The first section contains an introduction to the topic of complex non-split supermanifolds fixing the notation. Details can be found e.g. in [5] and [8].

Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a complex supermanifold with underlying complex manifold M , sheaf of superfunctions $\mathcal{O}_{\mathcal{M}}$ and projection onto numerical holomorphic functions $pr : \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{O}_M$. Setting $\mathcal{O}_{\mathcal{M}}^{nil} := \text{Ker}(pr)$ the sheaf $\mathcal{O}_{\mathcal{M}}^{nil}/(\mathcal{O}_{\mathcal{M}}^{nil})^2$ defines a holomorphic vector bundle E on M . Denote its automorphisms by $\text{Aut}(E)$, its sheaf of sections by \mathcal{O}_E , its full exterior power by ΛE and the sheaf of automorphisms of algebras on $\mathcal{O}_{\Lambda E}$ preserving the $\mathbb{Z}/2\mathbb{Z}$ -grading (but not necessarily the \mathcal{O}_M -module structure) by $\text{Aut}(\Lambda E)$. The rank of E is the odd dimension of \mathcal{M} . Following [5] denote by $G_E \subset \text{Aut}(\Lambda E)$ the subsheaf of groups given by elements $\varphi \in \text{Aut}(\Lambda E)$ satisfying

$$(\varphi - Id)(\mathcal{O}_{\Lambda^j E}) \subset \bigoplus_{k \geq 1} \mathcal{O}_{\Lambda^{j+2k} E} \quad \forall j \geq 0 .$$

It is proved in [5] that the isomorphism classes of complex supermanifolds associated with a given vector bundle $E \rightarrow M$ are in 1 : 1 correspondence to the $H^0(M, \text{Aut}(E))$ -orbits by conjugation on the Čech cohomology $H^1(M, G_E)$. Note that this cohomology is meant with respect to composition of maps with identity as neutral element. So $H^1(M, G_E)$ is nothing more but a pointed set. The orbit of the identity in $H^1(M, G_E)$ corresponds to the unique split supermanifold structure associated with $E \rightarrow M$ given by $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\Lambda E}$.

Following [8] let $\text{Der}^{(2)}(\Lambda E)$ denote the sheaf of even derivations on the sheaf of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras $\mathcal{O}_{\Lambda E}$ satisfying

$$w(\mathcal{O}_{\Lambda^j E}) \subset \bigoplus_{k \geq 1} \mathcal{O}_{\Lambda^{j+2k} E} \quad \forall j \geq 0 .$$

It is shown in [8] that the exponential map maps $\text{Der}^{(2)}(\Lambda E)$ isomorphically onto G_E . The sheaf $\text{Der}^{(2)}(\Lambda E)$ itself decomposes

$$\text{Der}^{(2)}(\Lambda E) = \bigoplus_{k=1}^{\infty} \text{Der}_{2k}(\Lambda E) , \tag{1}$$

where $Der_{2k}(\Lambda E)$ is the sheaf of even derivations satisfying $w(\mathcal{O}_{\Lambda^j E}) \subset \mathcal{O}_{\Lambda E^{j+2k}}$ for all $j \geq 0$. The appropriate cohomology on $Der^{(2)}(\Lambda E)$ is the usual abelian Čech cohomology with respect to the \mathcal{O}_M -module structure.

Remark 1. (1) In odd dimension 0 and 1 the sheaf $Der^{(2)}(\Lambda E)$ is trivial and hence there are only split supermanifold structures.

(2) In odd dimension 2 and 3 it is $Der^{(2)}(\Lambda E) = Der_2(\Lambda E)$. The exponential mapping is just adding the identity and the composition in $Der_2(\Lambda E)$ is zero. Hence the supermanifold structures on M associated with E correspond to the orbits of $H^0(M, Aut(E))$ by conjugation on $H^1(M, Der_2(\Lambda E))$.

(3) In odd dimension ≥ 4 the cohomologies $H^1(M, G_E)$ and $H^1(M, Der^{(2)}(\Lambda E))$ are in general not isomorphic any more.

In the following \mathbf{d} denotes the coboundary operator of the non-abelian cochain complex of G_E , while d denotes the respective operator for the abelian complex of $Der^{(2)}(\Lambda E)$.

2 Non-abelian cohomology in odd dimension 4 and 5

The non-abelian cohomology $H^1(M, G_E)$ for E of rank 4 or 5, is identified with a subset of the abelian cohomology $H^1(M, Der^{(2)}(\Lambda E))$.

In this section fix $rk(E) \in \{4, 5\}$. For a cocycle $\exp(u_2 + u_4) \in Z^1(M, G_E)$ where $u_{2s} \in C^1(M, Der_{2s}(\Lambda E))$, it is by direct calculation necessary that $u_2 \in Z^1(M, Der_2(\Lambda E))$. Furthermore define

$$c_{u_2} = pr_{End_4(\Lambda E)}(\mathbf{d} \exp(u_2)) \in C^2(M, End_4(\Lambda E)) ,$$

where the notion of $End^{(2)}(\Lambda E) = \bigoplus_{k=1}^{\infty} End_{2k}(\Lambda E)$ in the sheaf of complex linear endomorphisms of $\mathcal{O}_{\Lambda E}$ is defined analogously to (1). From the cocycle condition on $\exp(u_2 + u_4)$ it follows that $c_{u_2} = -du_4$ and hence c_{u_2} is a coboundary of derivations. Denote:

$$\tilde{Z}^1(M, Der_2(\Lambda E)) := \{u_2 \in Z^1(M, Der_2(\Lambda E)) \mid c_{u_2} \in B^2(M, Der_4(\Lambda E))\}$$

For later application note that for cochains $u_2 + u_4 \in C^1(M, Der^{(2)}(\Lambda E))$ and $v_2 + v_4 \in C^0(M, Der^{(2)}(\Lambda E))$ it follows by direct calculation that:¹

$$\exp(v_2 + v_4) \cdot \exp(u_2 + u_4) = \exp(u_2 + dv_2 + u_4 + dv_4 + F(u_2, v_2)) \quad (2)$$

$$\text{with } F(v_2, u_2) := \frac{1}{2}([v_{2,i} + v_{2,j}, u_{2,ij}] - [v_{2,i}, v_{2,j}])_{ij}$$

A map $D : H^0(M, Aut(E)) \times C^0(M, Der_2(\Lambda E)) \times \tilde{Z}^1(M, Der_2(\Lambda E)) \rightarrow C^1(M, Der_4(\Lambda E))$ is called compatible if it satisfies:

$$d(D(\varphi, v_2, u_2)) = c_{u_2} \quad (3)$$

$$\text{and } \varphi \cdot D(\varphi, v_2, u_2) = D(Id, 0, (\varphi \cdot u_2) + d(\varphi \cdot v_2)) + F(\varphi \cdot v_2, \varphi \cdot u_2) \quad (4)$$

¹Here $\exp(v_2 + v_4) \cdot \exp(u_2 + u_4)$ denotes $(\exp(v_{2,i} + v_{4,i}) \exp(u_{2,ij} + u_{4,ij}) \exp(-v_{2,j} - v_{4,j}))_{ij}$.

for all $\varphi \in H^0(M, \text{Aut}(E))$, $v_2 \in C^0(M, \text{Der}_2(\Lambda E))$ and $u_2 \in \tilde{Z}^1(M, \text{Der}_2(\Lambda E))$. It is called strongly compatible if additionally D satisfies

$$\varphi.D(\varphi, v_2, u_2) = D(\text{Id}, 0, \varphi.u_2)$$

for all allowed (φ, v_2, u_2) .

Lemma 1. *A compatible map D always exists. If $H^0(M, \text{Der}_2(\Lambda E)) = 0$, D can be chosen to be strongly compatible.*

Proof. Let $D(\text{Id}, 0, \cdot) : \tilde{Z}^1(M, \text{Der}_2(\Lambda E)) \rightarrow C^1(M, \text{Der}_4(\Lambda E))$ be any map satisfying (3) for the third argument. This exists due to the definition of $\tilde{Z}^1(M, \text{Der}_2(\Lambda E))$. Continue it via (4) to $H^0(M, \text{Aut}(E)) \times C^0(M, \text{Der}_2(\Lambda E)) \times \tilde{Z}^1(M, \text{Der}_2(\Lambda E))$. From (2) setting $u_4 = v_4 = 0$ it follows by direct calculation that

$$c_{u_2} = c_{u_2+dv_2} + dF(v_2, u_2) \quad (5)$$

for all $v_2 \in C^0(M, \text{Der}_2(\Lambda E))$ and $u_2 \in \tilde{Z}^1(M, \text{Der}_2(\Lambda E))$. Using this and (4), (3) holds for $\varphi = \text{Id}$. Since $u_2 \mapsto c_{u_2}$, d and F are $H^0(M, \text{Aut}(E))$ -equivariant, (3) holds for the continued map D .

In the case $H^0(M, \text{Der}_2(\Lambda E)) = 0$ we choose $D(\text{Id}, 0, \cdot)$ as above but require

$$D(\text{Id}, 0, (\varphi.u_2) + d(\varphi.v_2)) = D(\text{Id}, 0, \varphi.u_2) - F(\varphi.v_2, \varphi.u_2) \quad (6)$$

for all allowed (φ, v_2, u_2) . This can be satisfied due to (5) and since $d(\varphi.v_2) = 0$ implies $\varphi.v_2 = 0$ and so $F(\varphi.v_2, \varphi.u_2) = 0$. Now proceed as above to obtain $D(\varphi, v_2, u_2)$. We find with (4) that the additional requirement yields strong compatibility of D . \square

In the following let $H^0(M, \text{Der}_2(\Lambda E)) = 0$ and D will be chosen to be strongly compatible.

Lemma 2. *A strongly compatible D induces a well-defined map*

$$\sigma_D : H^1(M, G_E) \rightarrow H^1(M, \text{Der}_2(\Lambda E)) \oplus H^1(M, \text{Der}_4(\Lambda E))$$

given by $\sigma_D([\exp(u_2 + u_4)]) = ([u_2], [D(\text{Id}, 0, u_2) + u_4])$. If D additionally satisfies

$$\varphi.D(\text{Id}, 0, u_2) - D(\text{Id}, 0, \varphi.u_2) \in B^1(M, \text{Der}_4(\Lambda E)) \quad (7)$$

for all allowed (φ, u_2) then the map σ_D is $H^0(M, \text{Aut}(E))$ -equivariant.

Proof. Note that if $\exp(u_2 + u_4) \in Z^1(M, G_E)$ then $D(\varphi, v_2, u_2) + u_4 \in Z^1(M, \text{Der}_4(\Lambda E))$. Further it is with (2) and $H^0(M, \text{Aut}(E))$ -equivariance of \exp , d and F :

$$\begin{aligned} \sigma_D([\varphi.(\exp(v_2 + v_4). \exp(u_2 + u_4))]) \\ = ([(\varphi.u_2) + d(\varphi.v_2)], [D(\text{Id}, 0, (\varphi.u_2) + d(\varphi.v_2)) + (\varphi.u_4) + d(\varphi.v_4) + F(\varphi.v_2, \varphi.u_2)]) \end{aligned}$$

The representing elements differ from those of $\varphi.\sigma_D(\exp(u_2 + u_4))$ via equation (4) only by $(d(\varphi.v_2), D(\text{Id}, 0, \varphi.u_2) - \varphi.D(\text{Id}, 0, u_2) + d(\varphi.v_4))$. So σ_D is well-defined for $\varphi = \text{Id}$ and under the additional requirement on D also $H^0(M, \text{Aut}(E))$ -equivariant. \square

Note that σ_D is a bijection onto $\left(\tilde{Z}^1(M, \text{Der}_2(\Lambda E)) / B^1(M, \text{Der}_2(\Lambda E)) \right) \oplus H^1(M, \text{Der}_4(\Lambda E))$, the first summand being well-defined by (5). So from Lemmas 1 and 2 we can conclude:

Proposition 1. *In the case $H^0(M, \text{Der}_2(\Lambda E)) = 0$, there always exists a bijection:*

$$\sigma_D : H^1(M, G_E) \rightarrow \left(\tilde{Z}^1(M, \text{Der}_2(\Lambda E)) / B^1(M, \text{Der}_2(\Lambda E)) \right) \oplus H^1(M, \text{Der}_4(\Lambda E))$$

If a strongly compatible D satisfying (7) exists then σ_D is $H^0(M, \text{Aut}(E))$ -equivariant.

3 Cohomology for decomposable vector bundles

Assume in this section that $F \subset E$ is a complex sub vector bundle of rank ≤ 3 with $E = F \oplus F'$ as vector bundles. In the first part of this section E may have any rank $\geq rk(F)$ and the projection morphism $pr_F : E \rightarrow F$ is extended to $\Lambda pr_F : \mathcal{O}_{\Lambda E} \rightarrow \mathcal{O}_{\Lambda F}$. The goal is a restriction of a supermanifold structure on E to a supermanifold structure on F , and secondly expressing the cohomological data in the case $rk(F') \leq 2$ in terms of abelian cohomologies compatible with the decomposition.

Let $[\alpha] \in H^1(M, G_E)$ be a cohomology class represented by $\alpha \in Z^1(M, G_E)$. Denoting by $End(\Lambda F)$ the sheaf of endomorphisms of the sheaf of complex vector spaces $\mathcal{O}_{\Lambda F}$ define

$$\alpha_F := \Lambda pr_F \circ \alpha|_{\Lambda F} \in C^1(M, End(\Lambda F)) .$$

This cochain induces a supermanifold structure associated with F :

Lemma 3. *The cochain α_F lies in $Z^1(M, G_F)$ and the map*

$$H^1(M, G_E) \longrightarrow H^1(M, G_F), \quad [\alpha] \longmapsto [\alpha_F]$$

is well-defined.

Proof. Writing α_{ij} as the exponential of $\sum_{k=1}^{\infty} u_{2k,ij}$ with $u_{2k,ij} \in Der(M, Der_{2k}(\Lambda E))$ yields $(\alpha_F)_{ij}$ as the exponential of $\Lambda pr_F \circ u_{2,ij}|_{\Lambda F}$ since the $Der_4(\Lambda F)$ -term vanishes. Hence $(\mathbf{d}(\alpha_F))_{ijk} = \Lambda pr_F \circ (Id + u_{2,ij} + u_{2,jk} - u_{2,ik})|_{\Lambda F} = \Lambda pr_F \circ (\mathbf{d}(\alpha))_{ijk}|_{\Lambda F} = 0$. In a similar way it is obtained that the map $\alpha \rightarrow \alpha_F$ maps coboundaries to coboundaries. \square

Remark 2. (1) Note that α_F in general does not define the structure of a subsupermanifold. (2) It was used that the composition of endomorphisms, that increase the degree by 2, is zero up to odd dimension 3. The Lemma does in general not hold for higher rank subbundles. (3) In the special case² that the cocycle α can be chosen such that $\log(\alpha) \in Der^{(2\ell)}(\Lambda E)$, the Lemma follows for subbundles up to rank $4\ell - 1$.

Approaching odd dimension 4 and 5, from now on assume the case $E = F \oplus F'$ with $rk(F) \leq 3$ and $rk(F') \leq 2$. This yields a decomposition

$$\Lambda E = X \oplus Y \oplus Z, \quad \text{with } X = \Lambda F, \ Y = \Lambda F \otimes F', \ Z = \Lambda F \otimes \Lambda^2 F'$$

Denote for $S, T \in \{X, Y, Z\}$ by $Hom(T, S)$ the sheaf of homomorphisms of sheaves of complex vector spaces from \mathcal{O}_T to \mathcal{O}_S and set $End(T) := Hom(T, T)$ and $\tilde{G}_T := \exp(End_2(T))$. For a cochain $\alpha \in C^1(M, G_E)$ regard the cochains

$$\begin{aligned} \alpha_T &:= (pr_T \circ \alpha|_T) \in C^1(M, \tilde{G}_T) \quad \text{for } T = X, Y, Z, \\ u_{2,XY} + u_{4,XY} &:= (pr_Y \circ \alpha|_X) \in C^1(M, Der_2(X, Y)) \oplus C^1(M, Der_4(X, Y)) , \\ u_{2,XZ} + u_{4,XZ} &:= (pr_Z \circ \alpha|_X) \in C^1(M, Der_2(X, Z)) \oplus C^1(M, Der_4(X, Z)) , \\ u_{2,YZ} + u_{4,YZ} &:= (pr_Z \circ \alpha|_Y) \in C^1(M, Hom_2(Y, Z)) \oplus C^1(M, Hom_4(Y, Z)) , \\ u_{2,YX} &:= (pr_X \circ \alpha|_Y) \in C^1(M, Hom_2(Y, X)) , \\ u_{2,ZY} &:= (pr_Y \circ \alpha|_Z) \in C^1(M, Hom_2(Z, Y)) . \end{aligned} \tag{8}$$

²This case was pointed out to be of special interest in [8].

Note that the term $(pr_X \circ \alpha|_Z)$ missing in the list, vanishes for reasons of degree. All eleven mentioned cochain complexes, those of the first line with respect to composition, the remaining with respect to the sum of maps, are abelian. Continuing all eleven cochains by zero on the complement of their domain of definition respectively, their sum equals α . It follows from Proposition 1 and arguments similar to those in the proof of Lemma 3:

Proposition 2. *Let M be a complex manifold endowed with the sum of a rank ≤ 3 vector bundle F and a rank ≤ 2 vector bundle F' denoted $E = F \oplus F'$ with $H^0(M, \text{Der}_2(\Lambda E)) = 0$. Fix a strongly compatible map D as in section 2 and decompose $D = D_{XY} + D_{XZ} + D_{YZ}$ with $D_{PQ}(\varphi, v_2, u_2) := pr_Q \circ D(\varphi, v_2, u_2)|_P$. The map of cochains*

$$\alpha = \exp(u) \mapsto \left(\alpha_T, u_{2,RS}, D_{PQ}(Id, 0, u_2) + u_{4,PQ} \mid \begin{array}{l} R,S,T \in \{X,Y,Z\}, R \neq S, (R,S) \neq (Z,X) \\ (P,Q) \in \{(X,Y), (X,Z), (Y,Z)\} \end{array} \right)$$

induces a map of cohomologies from $H^1(M, G_E)$ to the direct sum $\oplus H$ of the eleven abelian cohomologies of the cochain complexes in (8). The induced map yields a bijection between $H^1(M, G_E)$ and the subset of elements in $\oplus H$ that can be represented by cocycles of the type $(\hat{\alpha}_T, \hat{u}_{2,RS}, \hat{u}_{4,PQ})$ satisfying $\hat{u}_2 := \sum_T \log(\hat{\alpha}_T) + \sum_{R,S} \hat{u}_{2,RS} \in Z^1(M, \text{Der}_2(\Lambda E))$ and $\hat{u}_4 := \sum_{R,S} \hat{u}_{4,RS} \in Z^1(M, \text{Der}_4(\Lambda E))$ as well as $c_{\hat{u}_2} \in B^2(M, \text{Der}_4(\Lambda E))$.

Remark 3. *If D also satisfies (7) then the inclusion $H^1(M, G_E) \hookrightarrow \oplus H$ is by Proposition 1 equivariant under the action of $H^0(M, \text{Aut}(F) \times \text{Aut}(F')) \subset H^0(M, \text{Aut}(E))$ acting diagonally on $\oplus H$.*

For the case of $rk(F') = 1$ denoting the line bundle F' by L , the result of Proposition 2 can be simplified. Most of the cochains in (8) vanish. The remaining are:

$$\alpha_F := \alpha_X, \quad \alpha_L := \alpha_Y, \quad u_F := u_{2,YX} \quad \text{and} \quad u_L = u_{2,L} + u_{4,L} := u_{2,XY} + u_{4,XY}$$

Note that for $f \in \mathcal{O}_M$, $s \in \mathcal{O}_L$ it is $u_F(fs) = pr_X((\alpha_F + u_L)(f)s + f(\alpha_L + u_F)(s)) = fu_F(s)$. Hence we can replace $H^1(M, \text{Hom}_2(Y, X))$ by $H^1(M, \text{Hom}_{\mathcal{O}_M}(L, \Lambda^3 F))$. Further we replace the cohomology $H^1(M, \text{Der}_4(X, Y))$ by $H^1(M, \text{Der}(\mathcal{O}_M) \otimes \mathcal{O}_{\Lambda^3 F \otimes L})$.

Corollary 1. *For a complex manifold M and the sum of a rank ≤ 3 vector bundle F and a line bundle L denoted $E = F \oplus L$ such that $H^0(M, \text{Der}_2(\Lambda E)) = 0$, fix a strongly compatible D . Then the elements $[\alpha]$ in the cohomology $H^1(M, G_E)$ correspond bijectively to the well defined classes*

$$([\alpha_F], [\alpha_L], [u_F], [u_{2,L}], [D(Id, 0, u_2) + u_{4,L}]) \in H^1(M, G_F) \oplus H^1(M, \tilde{G}_{\Lambda F \otimes L}) \\ \oplus H^1(M, \text{Hom}_{\mathcal{O}_M}(L, \Lambda^3 F)) \oplus H^1(M, \text{Der}_2(\Lambda F, \Lambda F \otimes L)) \oplus H^1(M, \text{Der}(\mathcal{O}_M) \otimes \mathcal{O}_{\Lambda^3 F \otimes L})$$

satisfying the two properties $u_2 := \alpha_F + \alpha_L + u_F + u_{2,L} - Id_{\Lambda E} \in Z^1(M, \text{Der}_2(\Lambda E))$ and $c_{u_2} \in B^2(M, \text{Der}(\mathcal{O}_M) \otimes \mathcal{O}_{\Lambda^3 F \otimes L})$.

Remark 4. *The condition $u_2 \in Z^1(M, \text{Der}_2(\Lambda E))$ contains that α_L is well-defined by α_F , u_F and $u_{2,L}$ and a term in $Z^1(M, \text{Hom}(L, \Lambda^2 F \otimes L))$. In particular for $f \in \mathcal{O}_M$, $s \in \mathcal{O}_L$ it is $\alpha_L(fs) = pr_Y((\alpha_F + u_L)(f)s + f(\alpha_L + u_F)(s)) = \alpha_F(f)s + f\alpha_L(s)$. So fixing α_F , two possible choices of α_L differ by an element in $Z^1(M, \text{Hom}_{\mathcal{O}_M}(L, \Lambda^2 F \otimes L))$. This allows to regard the freedom in $H^1(M, \tilde{G}_{\Lambda F \otimes L})$ as a freedom in $H^1(M, \text{Hom}_{\mathcal{O}_M}(L, \Lambda^2 F \otimes L))$.*

4 Examples on $\mathbb{P}^1(\mathbb{C})$

We discuss the orbit structure on $H^1(M, G_E)$ with respect to a maximal compact subgroup of $H^0(M, \text{Aut}(E))$ for the underlying manifold $M = \mathbb{P}^1(\mathbb{C})$ for a large class of vector bundles. The example is studied here using Proposition 1 and Corollary 1. We start with some general technical details and additional notation.

Let $\mathcal{O}(k)$ for $k \in \mathbb{Z}$ denote the line bundle on $\mathbb{P}^1(\mathbb{C})$ with divisor $k \cdot [0 : 1]$.³ Fixing the coordinate chart $\mathbb{P}^1(\mathbb{C}) \setminus \{[1 : 0]\} \rightarrow \mathbb{C}$, $[z_0 : z_1] \mapsto z := \frac{z_0}{z_1}$ and denoting by $\mathbb{C}[z]_{\leq l}$ the polynomials of degree $\leq l$ and $\{0\}$ if $l < 0$, we can identify the complex vector spaces $H^0(M, \mathcal{O}(k)) \cong \mathbb{C}[z]_{\leq k}$ and $H^1(M, \mathcal{O}(k)) \cong \frac{1}{z}\mathbb{C}[\frac{1}{z}]_{\leq -k-2}$. Further note that $\text{Hom}_{\mathcal{O}_M}(\mathcal{O}(i), \mathcal{O}(j)) \cong \mathcal{O}(j-i)$. Any complex vector bundle on $\mathbb{P}^1(\mathbb{C})$ can be decomposed into a direct sum of line bundles (see [6]) which are each isomorphic to one of the $\mathcal{O}(k)$. For a given vector bundle $E \rightarrow M$ fix such a decomposition $\mathcal{O}_E = \bigoplus_{i=1}^m \mathcal{O}(l_i)$ with $l_i \in \mathbb{Z}$. We fix $l_1 \leq \dots \leq l_m$. For later considerations we fix the standard local frame ξ_i for the standard bundle chart of $\mathcal{O}(l_i)$ on $\mathbb{P}^1(\mathbb{C}) \setminus \{[1 : 0]\}$ and denote by $(\frac{\partial}{\partial \xi_i})_i$ the dual frame to the local frame $(\xi_i)_i$ of E . Elements in $\text{Der}(\Lambda E)$ hence locally appear as linear combinations of terms $\xi^I f \frac{\partial}{\partial z}, \xi^I f \frac{\partial}{\partial \xi_i}$ with holomorphic numerical f and $I \in \{0, 1\}^m$.

Set $m_i = \#\{l_j \mid l_j = i\}$ and note that $H^0(M, \text{Hom}(\mathcal{O}(i), \mathcal{O}(j))) \cong H^0(M, \mathcal{O}(j-i))$ realized in the above coordinates by multiplication of polynomials $\mathbb{C}[z]_{\leq j-i} \times \mathbb{C}[z]_{\leq i} \rightarrow \mathbb{C}[z]_{\leq j}$. The global sections in $\text{Aut}(E)$ decompose into a semidirect product of groups $H^0(M, \text{Aut}(E)) \cong A(E) \ltimes N(E)$ with:

$$\begin{aligned} A(E) &:= X_{i=-\infty}^{\infty} GL(m_i, \mathbb{C}) \cong X_{i=-\infty}^{\infty} H^0(M, \text{Aut}(\mathcal{O}(i)^{m_i})) \\ N(E) &:= \text{Id}_{\mathcal{O}_{\Lambda E}} + \bigoplus_{i < j} (\text{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{m_j}) \otimes \mathbb{C}[z]_{\leq j-i}) \\ &\cong \text{Id}_{\mathcal{O}_{\Lambda E}} + \bigoplus_{i < j} H^0(M, \text{Hom}(\mathcal{O}(i)^{m_i}, \mathcal{O}(j)^{m_j})) \end{aligned}$$

Set $U(E) := X_{i=-\infty}^{\infty} U(m_i) \subset A(E)$. Denote by ρ and ρ^* the standard, resp. dual (here inverse transposed) action of the group $H^0(M, \text{Aut}(E))$, resp. of subgroups, on the vector space $\bigoplus_{i=-\infty}^{\infty} \mathbb{C}^{m_i}$.

Note further that for fixed $2k$ a derivation in $\text{Der}_{2k}(\Lambda E)$ is given by its values on the sections in the subbundle $\Lambda^0 E \oplus \Lambda^1 E \subset \Lambda E$. The continuation of a homomorphism on $\Lambda^1 E \subset \Lambda E$ by Leibniz rule and trivially on \mathcal{O}_M , respectively the restriction of a derivation to $\Lambda^0 E$ yield an exact sequence (see [4])

$$0 \rightarrow \text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E) \rightarrow \text{Der}_{2k}(\Lambda E) \rightarrow \text{Der}(\Lambda^0 E, \Lambda^{2k} E) \rightarrow 0$$

with $\text{Der}(\Lambda^0 E, \Lambda^{2k} E) = \text{Der}(\mathcal{O}_M) \otimes \mathcal{O}_{\Lambda^{2k} E}$. The long exact sequence of cohomology yields:

$$\begin{aligned} \dots \rightarrow H^0(M, \text{Der}(\Lambda^0 E, \Lambda^{2k} E)) &\rightarrow H^1(M, \text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) \rightarrow H^1(M, \text{Der}_{2k}(\Lambda E)) \\ &\rightarrow H^1(M, \text{Der}(\Lambda^0 E, \Lambda^{2k} E)) \rightarrow H^2(M, \text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) \rightarrow \dots \end{aligned}$$

In any case $H^2(M, \text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) = 0$ for reasons of the dimension. In this chapter we specialize on the case $H^0(M, \text{Der}(\Lambda^0 E, \Lambda^{2k} E)) = H^0(M, \text{Der}(\mathcal{O}_M) \otimes \mathcal{O}_{\Lambda^{2k} E}) = 0$. Due to

³Note that the sign convention here is opposite to the convention used e.g. in [10].

$Der(\mathcal{O}_M) = \mathcal{O}(2)$ this means $l_{m-1} + l_m < -2$. All appearing sheaves are coherent sheaves on the compact complex manifold M so we obtain an exact sequence of finite dimensional complex vector spaces:

$$0 \rightarrow H^1(M, Hom_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) \rightarrow H^1(M, Der_{2k}(\Lambda E)) \rightarrow H^1(M, Der(\Lambda^0 E, \Lambda^{2k} E)) \rightarrow 0$$

Note that in the case $H^1(M, Hom_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) = H^1(M, Der(\Lambda^0 E, \Lambda^{2k} E)) = \{0\}$, triviality of $H^1(M, Der_{2k}(\Lambda E))$ follows. Furthermore the above sequence is equivariant under the respective $H^0(M, Aut(E))$ -actions. There is a metric invariant under the maximal compact Lie subgroup $U(E)$ of $A(E)$, on the finite dimensional vector space $H^1(M, Der_{2k}(\Lambda E))$. The orthogonal complement to $H^1(M, Hom_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) \subset H^1(M, Der_{2k}(\Lambda E))$ yields a $U(E)$ -equivariant splitting:

$$H^1(M, Der_{2k}(\Lambda E)) \cong H^1(M, Der(\Lambda^0 E, \Lambda^{2k} E)) \oplus H^1(M, Hom_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) \quad (9)$$

We will use this splitting in the following.

In order to apply Proposition 1, we need $H^0(M, Der_2(\Lambda E)) = 0$. Note that the above long exact sequence of cohomology also yields exactness of:

$$H^0(M, Hom_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)) \rightarrow H^0(M, Der_{2k}(\Lambda E)) \rightarrow H^0(M, Der(\Lambda^0 E, \Lambda^{2k} E)) \quad (10)$$

Now for $k \geq 1$:

$$Hom_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E) \cong \mathcal{O}_{\Lambda^{2k+1} E} \otimes \mathcal{O}_E^* \quad \text{and} \quad Der(\Lambda^0 E, \Lambda^{2k} E) \cong \mathcal{O}_{\Lambda^{2k} E} \otimes \mathcal{O}(2) \quad (11)$$

So we assume in the following:

$$l_{m-1} + l_m < -2 \quad \text{and} \quad l_{m-2} + l_{m-1} + l_m - l_1 < 0$$

Then the global sections of the sheaves in (11) vanish and (10) yields $H^0(M, Der_2(\Lambda E)) = 0$.

Odd dimension 3

Assume now that $m = 3$. It is $H^1(M, G_E) \cong H^1(M, Der_2(\Lambda E))$. Using (9) and further that $Hom_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^3 E)$ consists of multiplication operators in $\mathcal{O}_{\Lambda^2 E}$, it is:

$$H^1(M, Der_2(\Lambda E)) \cong \bigoplus_{1 \leq i < j \leq 3} (H^1(M, \mathcal{O}(l_i + l_j + 2)) \oplus H^1(M, \mathcal{O}(l_i + l_j)))$$

Hence it is:

$$H^1(M, Der_2(\Lambda E)) \cong \bigoplus_{1 \leq i < j \leq 3} \left(\frac{1}{z} \mathbb{C} \left[\frac{1}{z} \right]_{\leq c_{ij}} \oplus \frac{1}{z} \mathbb{C} \left[\frac{1}{z} \right]_{\leq d_{ij}} \right) \quad \text{with} \quad (12)$$

$$c_{ij} := -l_i - l_j - 4 \quad \text{and} \quad d_{ij} := -l_i - l_j - 2$$

It follows that the group $U(E)$ acts on $H^1(M, Der_2(\Lambda E))$ by multiples of restrictions of the actions $\rho \wedge \rho$, resp. $\rho \wedge \rho \wedge \rho \otimes \rho^*$. Denote for the description of the $U(E)$ -orbits,

$$\mathbb{V}_k^n := \{U(k)(v) \mid v \in (\frac{1}{z} \mathbb{C} [\frac{1}{z}]_{\leq n})^k\}$$

for $0 \leq k, n$ and $\mathbb{V}_k^n = \{0\}$ else. Identify $\frac{1}{z} \mathbb{C} [\frac{1}{z}]_{\leq c_{ij}} \oplus \frac{1}{z} \mathbb{C} [\frac{1}{z}]_{\leq d_{ij}} \cong \frac{1}{z} \mathbb{C} [\frac{1}{z}]_{\leq c_{ij} + d_{ij} + 1}$ via the homomorphism $(p, q) \mapsto p + z^{-c_{ij}-1} q$.

Proposition 3. *The $U(E)$ -action on $H^1(M, \text{Der}_2(\Lambda E))$ has with respect to (12) the orbits:*

$$\mathbb{V}_3^{c_{12}+d_{12}+1} \text{ in the case } l_1 = l_2 = l_3,$$

$$\mathbb{V}_1^{c_{12}+d_{12}+1} \times \mathbb{V}_2^{c_{13}+d_{13}+1} \text{ in the case } l_1 = l_2 < l_3,$$

$$\mathbb{V}_2^{c_{12}+d_{12}+1} \times \mathbb{V}_1^{c_{23}+d_{23}+1} \text{ in the case } l_1 < l_2 = l_3,$$

$$\mathbb{V}_1^{c_{12}+d_{12}+1} \times \mathbb{V}_1^{c_{13}+d_{13}+1} \times \mathbb{V}_1^{c_{23}+d_{23}+1} \text{ in the case } l_1 < l_2 < l_3.$$

Proof. *Case $l_1 = l_2 = l_3$:* In this case it is $H^0(M, \text{Aut}(E)) \cong A(E) = GL(3, \mathbb{C})$. The action of $U(E) = U(3)$ on the vector space $H^1(M, G_E) \cong (T(c_{12}, d_{12}))^3$ with $T(c_{12}, d_{12}) := \frac{1}{z}\mathbb{C}[\frac{1}{z}]_{\leq c_{12}} \oplus \frac{1}{z}\mathbb{C}[\frac{1}{z}]_{\leq d_{12}}$ is given in a suitable basis by $A \mapsto M_A \otimes \text{Id}_{T(c_{12}, d_{12})}$, where M_A is the matrix of minors of A . Note further that it is $\{M_A \mid A \in U(3)\} = U(3)$.

Case $l_1 = l_2 < l_3$: In this case $U(E) = S^1 \times U(2)$. Its action on $H^1(M, G_E) \cong T(c_{12}, d_{12}) \oplus (T(c_{13}, d_{13}))^2$ is

$$(A, d) \mapsto (\det(A) \cdot \text{Id}_{T(c_{12}, d_{12})}) \oplus (d \cdot \det(A) \cdot (A^{-1})^T \otimes \text{Id}_{T(c_{13}, d_{13})})$$

yielding orbits parametrized by $\mathbb{V}_1^{c_{12}+d_{12}+1} \times \mathbb{V}_2^{c_{13}+d_{13}+1}$. *Case $l_1 < l_2 = l_3$* follows analogously. *Case $l_1 < l_2 < l_3$:* In this case it is $U(E) = (S^1)^3$. Its action on the identified vector space $H^1(M, G_E) \cong T(c_{12}, d_{12}) \oplus T(c_{13}, d_{13}) \oplus T(c_{23}, d_{23})$ is given by $(\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda_1 \cdot \lambda_2 \cdot \text{Id}_{T(c_{12}, d_{12})} \oplus \lambda_1 \cdot \lambda_3 \cdot \text{Id}_{T(c_{13}, d_{13})} \oplus \lambda_2 \cdot \lambda_3 \cdot \text{Id}_{T(c_{23}, d_{23})}$. \square

Odd dimension 4

In the case of $rk(E) = 4$, the \mathcal{O}_M -module $\text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^3 E)$ in (9) is generated by contractions in \mathcal{O}_{E^*} followed by a multiplication operator in $\mathcal{O}_{\Lambda^3 E}$. Furthermore note that $H^1(M, \text{Der}_4(\Lambda E)) \cong H^1(M, \text{Der}(\mathcal{O}_M) \otimes \mathcal{O}_{\Lambda^4 E})$. It follows:

Lemma 4. *The group $U(E)$ acts on $H^1(M, \text{Der}_2(\Lambda E))$ by multiples of restrictions of $\rho \wedge \rho$, resp. $\rho \wedge \rho \wedge \rho \otimes \rho^*$ according to the decomposition in (9). On $H^1(M, \text{Der}_4(\Lambda E))$ the group acts by the determinant.*

Note that $c_{u_2} = 0$ for all $u_2 \in Z^1(M, \text{Der}_2(\Lambda E))$ by the covering chosen above. So we have $\tilde{Z}^1(M, \text{Der}_2(\Lambda E)) = Z^1(M, \text{Der}_2(\Lambda E))$ on $\mathbb{P}^1(\mathbb{C})$. The canonical representatives χ of classes in $H^1(M, \text{Der}_2(\Lambda E))$ in the above coordinates are linear combinations of elements:

$$\begin{aligned} \frac{1}{z^{r+1}} \xi_i \xi_j \frac{\partial}{\partial z} & \quad \text{with } 0 \leq r \leq -l_i - l_j - 4 & \quad \text{and} \\ \frac{1}{z^{s+1}} \xi_i \xi_j \xi_k \frac{\partial}{\partial \xi_t} & \quad \text{with } 0 \leq s \leq -l_i - l_j - l_k + l_t - 2 \end{aligned}$$

Set $D(\text{Id}, 0, \chi) = 0$ for all of these canonical representatives. Then via (6), $D(\text{Id}, 0, \cdot)$ can be defined on all of $Z^1(M, \text{Der}_2(\Lambda E))$. Now the map D satisfies (7) for $\varphi \in U(E)$. Continue D to $U(E) \times C^0(M, \text{Der}_2(\Lambda E)) \times Z^1(M, \text{Der}_2(\Lambda E))$ as in the proof of Lemma 1.

Proposition 1 yields an $U(E)$ -equivariant bijection $\sigma_D : H^1(M, G_E) \rightarrow H^1(M, \text{Der}^{(2)}(\Lambda E))$ with the above D . We divide the general situation into three cases.

Fourfold sum of a line bundle

Assuming $\mathcal{O}_E = 4\mathcal{O}(l)$ it is with respect to the decomposition in (9):

$$\begin{aligned} H^1(M, \text{Der}_2(\Lambda E)) &\cong H^1(M, \mathcal{O}(2l+2))^6 \oplus H^1(M, \mathcal{O}(2l))^{16} \\ H^1(M, \text{Der}_4(\Lambda E)) &= H^1(M, \mathcal{O}(4l+2)) \end{aligned}$$

So it follows from $H^0(M, \text{Aut}(E)) \cong A(E)$, Proposition 1 and Lemma 4:

Theorem 1. *The supermanifold structures associated with E with $l < -1$ are parametrized by the $U(4)$ -orbits of the diagonal action given by $\rho \wedge \rho$, $\rho \wedge \rho \wedge \rho \otimes \rho^*$ and the determinant on the three summands of the vector space:*

$$\left(\frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]\right)_{\leq -2l-4}^6 \oplus \left(\frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]\right)_{\leq -2l-2}^{16} \oplus \frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]_{\leq -4l-4}$$

Two couples

Assume now that $\mathcal{O}_E = 2\mathcal{O}(l) \oplus 2\mathcal{O}(l')$ with $l < l'$. Then $A(E) = GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ and $N(E) = Id_E + Hom(\mathbb{C}^2, \mathbb{C}^2) \otimes \mathbb{C}[z]_{\leq l'-l}$ as lower-left block matrices. Denote the standard and determinant action of the factors of $A(E)$ by ρ_i , resp. det_i , $i = 1, 2$. It is with respect to the decomposition in (9) – each line one term:

$$\begin{aligned} H^1(M, \text{Der}_2(\Lambda E)) &\cong H^1(M, \mathcal{O}(2l+2)) \oplus (H^1(M, \mathcal{O}(l+l'+2)))^4 \oplus H^1(M, \mathcal{O}(2l'+2)) \\ &\quad \oplus H^1(M, \mathcal{O}(2l))^4 \oplus (H^1(M, \mathcal{O}(l+l')))^8 \oplus H^1(M, \mathcal{O}(2l'))^4 \\ H^1(M, \text{Der}_4(\Lambda E)) &\cong H^1(M, \mathcal{O}(2(l+l')+2)) \end{aligned}$$

Proposition 1 and Lemma 4 yield:

Proposition 4. *Non-split supermanifold structures on $\mathbb{P}^1(\mathbb{C})$ associated with a vector bundle of the form $\mathcal{O}_E = 2\mathcal{O}(l) \oplus 2\mathcal{O}(l')$ with $l < l'$ only appear if $l \leq -1$. Identifying in the case $l' < -1$ the cohomology $H^1(M, G_E)$ with*

$$\begin{aligned} &\frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]_{\leq -2l-4} \oplus \left(\frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]\right)_{\leq -l-l'-4}^4 \oplus \frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]_{\leq -2l'-4} \\ \oplus &\left(\frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]\right)_{\leq -2l-2}^4 \oplus \left(\frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]\right)_{\leq -l-l'-2}^8 \oplus \left(\frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]\right)_{\leq -2l'-2}^4 \oplus \frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]_{\leq -2l-2l'-4} \end{aligned}$$

the diagonal action of $U(E) = U(2) \times U(2)$ is given by det_1 , $\rho_1 \otimes \rho_2$, det_2 , $det_1 \cdot \rho_2 \otimes \rho_2^*$, $(det_1 \cdot \rho_2 \otimes \rho_1^*) \oplus (det_2 \cdot \rho_1 \otimes \rho_2^*)$, $det_2 \cdot \rho_1 \otimes \rho_1^*$ and $det_1 \cdot det_2$.

A distinct line bundle

Decompose a rank 4 vector bundle $E = F \oplus L$ with $\mathcal{O}_F = \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \mathcal{O}(l_3)$ ordered to $l_1 \leq l_2 \leq l_3$ and $\mathcal{O}_L = \mathcal{O}(l)$ with $l \neq l_i$ for all $i = 1, 2, 3$. Note that we can assume without loss of generality that $l < l_1$ or $l > l_3$. Following Corollary 1 the relevant cohomologies involved in a classification of supermanifold structures are $H^1(M, G_F)$ given analogue to the case of 3 odd dimensions in (12), $H^1(M, Hom_{\mathcal{O}_M}(L, \Lambda^3 F)) \cong H^1(M, \mathcal{O}(l_1 + l_2 + l_3 - l)) \cong \frac{1}{z}\mathbb{C}\left[\frac{1}{z}\right]_{\leq c}$ with

$$c := -l_1 - l_2 - l_3 + l - 2$$

and $H^1(M, \text{Der}_2(\Lambda F, \Lambda F \otimes L))$, $H^1(M, \text{Der}_4(\Lambda F, \Lambda F \otimes L))$ and $H^1(M, \tilde{G}_{\Lambda F \otimes L})$. By Remark 4 the only remaining relevant term in the last cohomology is generated by $\text{id}_{\mathcal{O}_L}$ followed by a multiplication in $\mathcal{O}_{\Lambda^2 F}$ yielding $\bigoplus_{1 \leq i, j \leq 3} H^1(M, \mathcal{O}(l_i + l_j)) \cong \bigoplus_{1 \leq i, j \leq 3} \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d'_{ij}}$ with

$$d'_{ij} := -l_i - l_j - 2.$$

It is with respect to the decomposition in (9) in the case $l_3 + l_4 < -2$, resp. $l_4 + l < -2$:

$$\begin{aligned} H^1(M, \text{Der}_2(\Lambda F, \Lambda F \otimes L)) \cong & \bigoplus_{1 \leq i \leq 3} \left(H^1(M, \mathcal{O}(l_i + l + 2)) \right. \\ & \left. \oplus H^1(M, \mathcal{O}(l_i + l))^2 \oplus H^1(M, \mathcal{O}(l_1 + l_2 + l_3 + l - 2l_i)) \right) \end{aligned}$$

So $H^1(M, \text{Der}_2(\Lambda F, \Lambda F \otimes L)) \cong \bigoplus_{1 \leq i \leq 3} \left(\frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq c_i} \oplus \left(\frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d_i} \right)^2 \oplus \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d'_i} \right)$ with:

$$c_i := -l_i - l - 4, \quad d_i := -l_i - l - 2, \quad d'_i := -l_1 - l_2 - l_3 - l + 2l_i - 2$$

Finally $H^1(M, \text{Der}_4(\Lambda F, \Lambda F \otimes L)) \cong H^1(M, \mathcal{O}(l_1 + l_2 + l_3 + l + 2)) \cong \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d}$ with

$$d := -l_1 - l_2 - l_3 - l - 4$$

It is $H^0(M, \text{Aut}(E)) \cong (A(F) \times A(L)) \ltimes (N(F) \times N'(E))$ with $A(F) \subset GL(3, \mathbb{C})$, $A(L) = \mathbb{C}^\times$, $N(F)$ as in the case of odd dimension three and $N'(E) = \text{pr}_{E \rightarrow L} + \bigoplus_{i=1}^3 \mathbb{C}[z]_{\leq |l_i - l|}$ as upper-right, resp. lower-left block matrices if $l < l_1 \leq l_2 \leq l_3$, resp. if $l_1 \leq l_2 \leq l_3 < l$.

Corollary 1, Proposition 3 and Lemma 4 yield:

Proposition 5. *Non-split supermanifold structures of odd dimension 4 on $\mathbb{P}^1(\mathbb{C})$ associated with a vector bundle $\mathcal{O}_E = \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \mathcal{O}(l_3) \oplus \mathcal{O}(l)$ with $l < l_1 \leq l_2 \leq l_3$, resp. $l_1 \leq l_2 \leq l_3 < l$ only appear if $l + l_1 \leq -2$, resp. $l_1 + l_2 \leq -2$. Identifying in the case $l_3 + l_4 < -2$, resp. $l_4 + l < -2$ the cohomology $H^1(M, G_E)$ with*

$$\begin{aligned} & \bigoplus_{1 \leq i, j \leq 3} \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq c_{ij}} \oplus \bigoplus_{1 \leq i \leq 3} \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq c_i} \oplus \bigoplus_{1 \leq i \leq 3} \left(\left(\frac{1}{z} \mathbb{C}[\frac{1}{z}]_{d_i} \right)^2 \oplus \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d'_i} \right) \\ \oplus & \bigoplus_{1 \leq i, j \leq 3} \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d_{ij}} \oplus \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq c} \oplus \bigoplus_{1 \leq i, j \leq 3} \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d'_{ij}} \oplus \frac{1}{z} \mathbb{C}[\frac{1}{z}]_{\leq d} \end{aligned}$$

the diagonal action of $U(F) \subset U(3)$ is given by the restrictions of $\rho \wedge \rho$ on the first and sixth, of ρ on the second, of $\rho \wedge \rho \otimes \rho^*$ on the third, of $\rho \wedge \rho \wedge \rho \otimes \rho^*$ on the fourth and by the determinant action on the fifth and seventh summand. The diagonal action of $A(L) = S^1$ is trivial on the first, fourth and sixth, dual on the fifth and standard on the three remaining summands.

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